§5.3 (p.127)

1. 次の変数変換に対する Jacobian を求めよ.

(1) $x = 2r\cos\theta,\ y = 5r\sin\theta.$

(2) $u = x^2 + y^2,\ v = xy.$

(3) $x = \cos u,\ y = \sin v \cos u.$

【解答】(1) $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 2\cos\theta & -2r\sin\theta \\ 5\sin\theta & 5r\cos\theta \end{bmatrix}, \quad \therefore \quad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = 10r.$

(2) $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}, \quad \therefore \quad \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 2(x^2 - y^2).$

(3) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} -\sin u & 0 \\ -\sin v \sin u & \cos v \cos u \end{bmatrix}, \quad \therefore \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = -\sin u \cos u \cos v.$ □

2. 次の 2 重積分の値を求めよ.

(1) $\displaystyle\iint_D xy dxdy, \qquad D = \{(x,y) \mid 2 \le 2x + 3y \le 4,\ -1 \le 5x + y \le 3\}.$

(2) $\displaystyle\iint_D (x-y)^2 dxdy, \qquad D = \{(x,y) \mid |x - 2y| \le 1,\ |x + y| \le 1\}.$

(3) $\displaystyle\iint_D \tan(x+y) dxdy, \qquad D = \{(x,y) \mid 0 \le x + y \le \frac{\pi}{4},\ 0 \le x - y \le \frac{\pi}{4}\}.$

【解答】(1) $u = 2x + 3y,\ v = 5x + y,\ E = \{(u,v) \mid 2 \le u \le 4,\ -1 \le v \le 3\}$ とすると $|\frac{\partial(u,v)}{\partial(x,y)}| = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = -13,$ $x = (-u + 3v)/13,\ y = (5u - 2v)/13$ より

$$\iint_D xy dxdy = \frac{1}{13^2}\iint_E (-u+3v)(5u-2v)\ \mathbf{abs}\left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right) dudv = \frac{1}{13^3}\int_2^4 \left(\int_{-1}^3 (-5u^2 + 17uv - 6v^2) dv\right) du$$
$$= \frac{1}{13^3}\int_2^4 (-20u^2 + 68u - 56) du = -\frac{232}{3 \cdot 13^3}$$

(2) $u = x - 2y,\ v = x + y,\ E = \{(u,v) \mid |u| \le 1,\ |v| \le 1\}$ とすると $|\frac{\partial(u,v)}{\partial(x,y)}| = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3,\ x = (u+2v)/3,\ y = (-u+v)/3$ より

$$\iint_D (x-y)^2 dxdy = \frac{1}{3^2}\iint_E (2u - v)^2\ \mathbf{abs}\left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right) dudv = \frac{1}{3^3}\int_{-1}^1 \left(\int_{-1}^1 (2u - v)^2 dv\right) du$$
$$= \frac{2}{3^4}\int_{-1}^1 (12u^2 + 1) du = \frac{4}{3^4}\int_0^1 (12u^2 + 1) du = \frac{20}{81}$$

(3) $u = x + y,\ v = x - y,\ E = \{(u,v) \mid 0 \le u \le \frac{\pi}{4},\ 0 \le v \le \frac{\pi}{4}\}$ とすると $|\frac{\partial(u,v)}{\partial(x,y)}| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2,\ x = (u+v)/2,\ y = (u - v)/2$ より

$$\iint_D \tan(x+y) dxdy = \iint_E \tan u\ \mathbf{abs}\left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right) dudv = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\int_0^{\frac{\pi}{4}} \tan u dv\right) du$$
$$= \frac{\pi}{8}\left[-\log\cos u\right]_0^{\frac{\pi}{4}} = \frac{\log 2}{16}\pi$$

□

3. 次の 2 重積分の値を変数変換を行って求めよ.

(1) $\displaystyle\iint_D (4x^2 - y^2) dxdy, \qquad D = \{(x,y) \mid 0 \le x,\ 1 \le xy \le 2,\ 2 \le 2x - y \le 3\}.$

(2) $\displaystyle\iint_D (3x^2 + 2y^2) dxdy, \qquad D = \{(x,y) \mid x^2 + y^2 \le 3x,\ 0 \le y\}.$

(3) $\displaystyle\iint_D \frac{x^2 - y^2}{x^2 + y^2 + 1} dxdy, \qquad D = \{(x,y) \mid 2 \le x^2 + y^2 \le 4,\ 0 \le y \le x\}.$

(4) $\displaystyle\iint_D \sin\left(\frac{x^2}{4}+\frac{y^2}{9}\right)dxdy, \qquad D=\left\{(x,y)\,\middle|\,\frac{x^2}{4}+\frac{y^2}{9}\le 1,\ 0\le x,\ 0\le y\right\}.$

(5) $\displaystyle\iint_D x dxdy, \qquad D=\{(x,y)\mid \sqrt{x}+\sqrt{y}\le 1,\ 0\le x,\ 0\le y\}.$

【解答】 求める積分を I と記す．

(1) $u=2x+y,\ v=2x-y\ \ (x=(u+v)/4,\ y=(u-v)/2)$ と置けば

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{bmatrix}1/4 & 1/4\\ 1/2 & -1/2\end{bmatrix}, \qquad \det\frac{\partial(x,y)}{\partial(u,v)}=-\frac{1}{4}$$

であり，D は

$$D'=\{(u,v)\mid u+v\ge 0,\ 8\le u^2-v^2\le 16,\ 2\le v\le 3\}=\{(u,v)\mid \sqrt{8+v^2}\le u\le\sqrt{16+v^2},\ 2\le v\le 3\}$$

に変換される．従って変数変換公式より

$$I=\iint_{D'} uv\left|-\frac{1}{4}\right|dudv=\frac{1}{4}\iint_{D'} uvdudv=\frac{1}{4}\int_2^3\left(\int_{\sqrt{8+v^2}}^{\sqrt{16+v^2}} uvdu\right)dv$$
$$=\frac{1}{4}\int_2^3 v\left[\frac{u^2}{2}\right]_{\sqrt{8+v^2}}^{\sqrt{16+v^2}}dv=\frac{1}{8}\int_2^3 v\{16+v^2-(8+v^2)\}dv=\int_2^3 vdv=\frac{5}{2}$$

(2) 極座標，即ち $x=r\cos\theta,\ y=r\sin\theta$ を用いれば，D は

$$D'=\{(r,\theta)\mid 0\le\theta\le\pi/2,\ 0\le r\le 3\cos\theta\}$$

に変換される．従って変数変換公式より

$$I=\iint_{D'}(3r^2\cos^2\theta+2r^2\sin^2\theta)|r|drd\theta=\int_0^{\pi/2}(\cos^2\theta+2)\left(\int_0^{3\cos\theta} r^3dr\right)d\theta$$
$$=\frac{81}{4}\left\{\int_0^{\pi/2}\cos^6\theta d\theta+2\int_0^{\pi/2}\cos^4\theta d\theta\right\}=\frac{81}{4}\left(\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)=\frac{1377}{128}\pi$$

(3) 極座標を用いれば D は

$$D'=\{(r,\theta)\mid\sqrt{2}\le r\le 2,\ 0\le\theta\le\pi/4\}$$

に変換される．従って変数変換公式より

$$I=\iint_{D'}\frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2+1}|r|drd\theta=\left(\int_{\sqrt{2}}^2\frac{r^3}{r^2+1}dr\right)\left(\int_0^{\pi/4}\cos 2\theta d\theta\right)$$
$$=\left[\frac{1}{2}(r^2-\log(r^2+1))\right]_{\sqrt{2}}^2\times\left[\frac{1}{2}\sin 2\theta\right]_0^{\pi/4}=\frac{1}{4}(2+\log 3-\log 5)$$

(4) $x=2r\cos\theta,\ y=3r\sin\theta$ と置けば

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{bmatrix}2\cos\theta & -2r\sin\theta\\ 3\sin\theta & 3r\cos\theta\end{bmatrix}, \qquad \det\frac{\partial(x,y)}{\partial(r,\theta)}=6r$$

であり，D は $D'=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le\pi/2\}$ に変換される．従って変数変換公式より

$$I=\iint_{D'}\sin r^2|6r|drd\theta=3\left(\int_0^1 2r\sin r^2dr\right)\left(\int_0^{\pi/2}d\theta\right)=\frac{3\pi}{2}\left[-\cos r^2\right]_0^{\pi/2}=\frac{3\pi}{2}\left(1-\cos\frac{\pi^2}{4}\right)$$

(5) $x=r\cos^4\theta,\ y=r\sin^4\theta$ と置けば

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{bmatrix}\cos^4\theta & -4r\cos^3\theta\sin\theta\\ \sin^4\theta & 4r\sin^3\theta\cos\theta\end{bmatrix}, \qquad \det\frac{\partial(x,y)}{\partial(r,\theta)}=4r\cos^3\theta\sin^3\theta$$

であり，D は $D'=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le\pi/2\}$ に変換される．従って変数変換公式より

$$I=\iint_{D'}r\cos^4\theta|4r\cos^3\theta\sin^3\theta|drd\theta=4\left(\int_0^1 r^2dr\right)\left(\int_0^{\pi/2}(1-\sin^2\theta)^3\sin^3\theta\cos\theta d\theta\right)$$
$$\underset{t=\sin\theta}{=}\frac{4}{3}\left(\int_0^1(1-t^2)^3t^3dt\right)\underset{s=t^2}{=}\frac{2}{3}\left(\int_0^1(1-s)^3sds\right)=\frac{1}{30}$$

□