**§5.2** (p.121)

**1.** 次の累次積分の値を求めよ．

(1) $\displaystyle\int_3^4\left(\int_1^2(x+y)dx\right)dy.$

(2) $\displaystyle\int_2^5\left(\int_1^y(x^2+y)dx\right)dy$

(3) $\displaystyle\int_2^3\left(\int_1^{2x}e^{x+y}dy\right)dx.$

(4) $\displaystyle\int_0^{\frac{\pi}{4}}\left(\int_0^{\frac{\pi}{4}}\cos(x+y)\sin 2ydx\right)dy$

(5) $\displaystyle\int_1^{\frac{\pi}{2}}\left(\int_0^{\frac{1}{y}}\frac{1}{1+x^2y^2}dx\right)dy$

(6) $\displaystyle\int_0^1\left(\int_{\frac{x}{2}}^x\cos\frac{\pi y}{x}dy\right)dx$

【解答】 以下，求める積分を $I$ と記す．

(1) $\displaystyle I=\int_3^4\left[\frac{x^2}{2}+yx\right]_1^2dy=\int_3^4\left(\frac{3}{2}+y\right)dy=\left[\frac{3}{2}y+\frac{y^2}{2}\right]_3^4=5.$

(2) $\displaystyle I=\int_2^5\left[\frac{x^3}{3}+yx\right]_1^ydy=\int_2^5\left(\frac{y^3}{3}+y^2-y-\frac{1}{3}\right)dy=\left[\frac{y^4}{12}+\frac{y^3}{3}-\frac{y^2}{2}-\frac{y}{3}\right]_2^5=\frac{313}{4}.$

(3) $\displaystyle I=\int_2^3\left[e^{x+y}\right]_1^{2x}dx=\int_2^3(e^{3x}-e^{x+1})dx=\left[\frac{e^{3x}}{3}-e^{x+1}\right]_2^3=\frac{e^9}{3}-\frac{e^6}{3}-e^4+e^3.$

(4)
$$\begin{aligned}I&=\int_0^{\frac{\pi}{4}}\sin 2y\left(\int_0^{\frac{\pi}{4}}\cos(x+y)dx\right)dy=\int_0^{\frac{\pi}{4}}\sin 2y\Big[\sin(x+y)\Big]_0^{\frac{\pi}{4}}dy\\&=\int_0^{\frac{\pi}{4}}\sin 2y(\sin(\frac{\pi}{4}+y)-\sin y)dy=\int_0^{\frac{\pi}{4}}2\sin y\cos y(\frac{1}{\sqrt{2}}\cos y+\frac{1}{\sqrt{2}}\sin y-\sin y)dy\\&=\sqrt{2}\int_0^{\frac{\pi}{4}}(-\cos^2 y(-\sin y)+(1-\sqrt{2})\sin^2 y\cos y)dy=\sqrt{2}\left[-\frac{1}{3}\cos^3 y+\frac{1-\sqrt{2}}{3}\sin^3 y\right]_0^{\frac{\pi}{4}}=\frac{\sqrt{2}}{6}\end{aligned}$$

(5) $\displaystyle I=\int_1^{\frac{\pi}{2}}\left(\int_0^{\frac{1}{y}}\frac{1}{1+x^2y^2}dx\right)dy=\int_1^{\frac{\pi}{2}}\left[\frac{1}{y}\tan^{-1}(xy)\right]_0^{\frac{1}{y}}dy=\frac{\pi}{4}\int_1^{\frac{\pi}{2}}\frac{1}{y}dy=\frac{\pi}{4}\Big[\log y\Big]_1^{\frac{\pi}{2}}=\frac{\pi}{4}\log\frac{\pi}{2}$

(6) $\displaystyle I=\int_0^1\left[\frac{x}{\pi}\sin\frac{\pi y}{x}\right]_{\frac{x}{2}}^x dx=-\frac{1}{\pi}\int_0^1 xdx=-\frac{1}{2\pi}$ □

**2.** 次の 2 重積分を累次積分に直し，積分の値を求めよ．

(1) $\displaystyle\iint_D(x^2y+yx+y^2)dxdy,\qquad D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 2\}.$

(2) $\displaystyle\iint_D\log(x^2+xy)dxdy,\qquad D=\{(x,y)\mid 1\le x\le 2,\ 3\le y\le 4\}.$

(3) $\displaystyle\iint_D xydxdy,\qquad D=\{(x,y)\mid x^2\le y\le x\}.$

(4) $\displaystyle\iint_D xdxdy,\qquad D=\{(x,y)\mid 0\le x,\ \frac{x^2}{4}+\frac{y^2}{9}\le 1\}.$

(5) $\displaystyle\iint_D(x^2+y^2)dxdy,\qquad D=\{(x,y)\mid 0\le x,\ 0\le y,\ x+y\le 2\}.$

(6) $\displaystyle\iint_D 2xydxdy,\qquad D=\{(x,y)\mid (x-2)^2\le y\le x\}.$

(7) $\displaystyle\iint_D\frac{1}{x^2+y^2}dxdy,\qquad D=\{(x,y)\mid 1\le x\le 2,\ 0\le y\le x\}.$

【解答】 以下，求める積分を $I$ と記す．尚，(3) ～ (6) の領域 $D$ は下の図参照．

(1) $\displaystyle I=\int_0^1\left(\int_0^2(x^2y+yx+y^2)dy\right)dx=\int_0^1\left(2x^2+2x+\frac{8}{3}\right)dx=\frac{13}{3}$

(2)

$$I=\int_1^2\left(\int_3^4(\log x+\log(x+y))dy\right)dx=\int_1^2\Big[y\log x+(x+y)\log(x+y)-(x+y)\Big]_{y=3}^{y=4}dx$$

$$=\int_1^2(\log x+(x+4)\log(x+4)-(x+3)\log(x+3)-1)dx$$

$$=\Big[x\log x-2x+\frac{(x+4)^2}{2}\log(x+4)-\frac{(x+4)^2}{4}-\frac{(x+3)^2}{2}\log(x+3)+\frac{(x+3)^2}{4}\Big]_1^2$$

$$=36\log 2+18\log 3-25\log 5-\frac{5}{2}$$

(3) $D=\{(x,y)\mid 0\le x\le 1,\ x^2\le y\le x\}$ だから

$$I=\int_0^1\left(\int_{x^2}^x xydy\right)dx=\int_0^1\left[\frac{xy^2}{2}\right]_{y=x^2}^{y=x}dx=\frac{1}{2}\int_0^1(x^3-x^5)dx=\frac{1}{24}$$

(4) $y^2\le 9(1-\frac{x^2}{4})\le 3^2$ より $D=\{(x,y)\mid -3\le y\le 3,\ 0\le x\le 2\sqrt{1-\frac{y^2}{9}}\}$ となる.

$$I=\int_{-3}^3\left(\int_0^{2\sqrt{1-\frac{y^2}{9}}}xdx\right)dy=\int_{-3}^3\left[\frac{x^2}{2}\right]_0^{2\sqrt{1-\frac{y^2}{9}}}dy=\frac{1}{2}\int_{-3}^3 4\left(1-\frac{y^2}{9}\right)dy=4\int_0^3\left(1-\frac{y^2}{9}\right)dy=8$$

(5) $D=\{(x,y)\mid 0\le x\le 2,\ 0\le y\le -x+2\}$ だから

$$I=\int_0^2\left(\int_0^{-x+2}(x^2+y^2)dy\right)dx=\int_0^2\left[x^2y+\frac{y^3}{3}\right]_0^{-x+2}dx=\int_0^2\left(-x^3+2x^2-\frac{(x-2)^3}{3}\right)dx=\frac{8}{3}$$

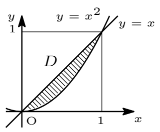
(6) $(x-2)^2=x\ \Leftrightarrow\ x=1,4$ より $D=\{(x,y)\mid 1\le x\le 4,\ (x-2)^2\le y\le x\}$.

$$I=\int_1^4\left(\int_{(x-2)^2}^x 2xydy\right)dx=\int_1^4\Big[xy^2\Big]_{(x-2)^2}^x dx=\int_1^4(x^3-(x-2)^5-2(x-2)^4)dx$$

$$=\Big[\frac{x^4}{4}-\frac{(x-2)^6}{6}-2\frac{(x-2)^5}{5}\Big]_1^4=\frac{801}{20}$$

(7) $I=\int_1^2\left(\int_0^x(x^2+y^2)dy\right)dx=\int_1^2\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx=\frac{\pi}{4}\int_1^2\frac{dx}{x}=\frac{\pi}{4}\log 2$

(3)

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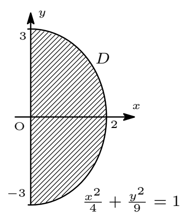


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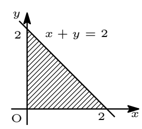


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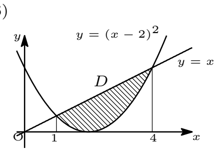



□

**3.** 次の累次積分から領域 $D$ を決め，2 重積分を用いて表せ．またこれを用いて積分順序を交換せよ．

(1) $\int_1^2\left(\int_0^1 f(x,y)dy\right)dx.$

(2) $\int_0^1\left(\int_0^x f(x,y)dy\right)dx$

(3) $\int_0^1\left(\int_0^{x^2} f(x,y)dy\right)dx.$

(4) $\int_0^1\left(\int_y^{\sqrt{y}} f(xmy)dx\right)dy$

(5) $\int_0^1\left(\int_{y^2}^{\sqrt{y}} f(x,y)dx\right)dy$

【解答】 以下，求める積分を $I$ と記す．

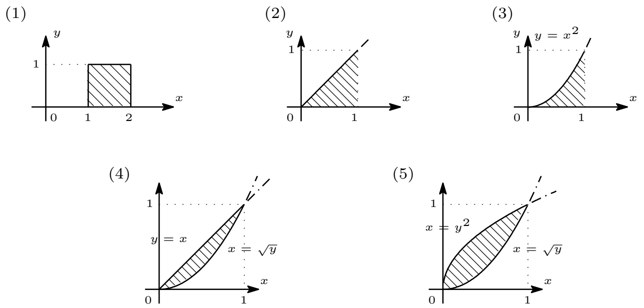
(1) $D=\{(x,y):1\le x\le 2,\ 0\le y\le 1\}$ $\quad I=\int_0^1\left(\int_1^2 f(x,y)dx\right)dy.$

(2) $D=\{(x,y):0\le x\le 1,\ 0\le y\le x\}$. $\quad I=\displaystyle\int_0^1\left(\int_y^1 f(x,y)dx\right)dy.$

(3) $D=\{(x,y):0\le x\le 1,\ 0\le y\le x^2\}$. $\quad I=\displaystyle\int_0^1\left(\int_{\sqrt{y}}^1 f(x,y)dx\right)dy.$

(4) $D=\{(x,y):0\le y\le 1,\ y\le x\le \sqrt{y}\}$. $\quad I=\displaystyle\int_0^1\left(\int_{x^2}^x f(x,y)dy\right)dx.$

(5) $D=\{(x,y):0\le y\le 1,\ y^2\le x\le \sqrt{y}\}$. $\quad I=\displaystyle\int_0^1\left(\int_{x^2}^{\sqrt{x}} f(x,y)dy\right)dx.$




□

**4.** 次の累次積分の値を求めよ.

(1) $\displaystyle\int_0^1\left(\int_0^1 \frac{2}{y^2+1}dy\right)dx.$ (2) $\displaystyle\int_{\frac{1}{2}}^2\left(\int_{\frac{1}{2}}^2 ye^{xy}dy\right)dx$

(3) $\displaystyle\int_{-1}^0\left(\int_0^{\sqrt{\frac{y+1}{2}}}\sqrt{y-x^2+1}dx\right)dy+\int_0^1\left(\int_{\sqrt{y}}^{\sqrt{\frac{y+1}{2}}}\sqrt{y-x^2+1}dx\right)dy$

**【解答】** 以下，求める積分を $I$ と記す.

(1) $I=2\displaystyle\int_0^1\Big[\tan^{-1}y\Big]_0^1dx=\frac{\pi}{2}.\quad I=\int_0^1\left(\int_0^1\frac{2}{y^2+1}dx\right)dy=\int_0^1\left[\frac{2x}{y^2+1}\right]_0^1dy=2\Big[\tan^{-1}y\Big]_0^1=\frac{\pi}{2}$

(2) このままでは積分出来ないので積分順序の変更を行う.

$$\begin{aligned}I&=\int_{\frac{1}{2}}^2\left(\int_{\frac{1}{2}}^2 ye^{xy}dx\right)dy=\int_{\frac{1}{2}}^2\Big[e^{xy}\Big]_{\frac{1}{2}}^2dy\\&=\int_{\frac{1}{2}}^2(e^{2x}-e^{\frac{x}{2}})dy=\left[\frac{e^{2x}}{2}-2e^{\frac{x}{2}}\right]_{\frac{1}{2}}^2=\frac{e^4}{2}-\frac{5}{2}e+2e^{\frac{1}{4}}\end{aligned}$$

※ **積分指数関数** $\displaystyle\int\frac{e^x}{x}dx$ は積分出来ない (=初等関数を用いて表現できない) ので，積分順序の変更が必要となる.

(3) $D=\left\{(x,y):-1\le y\le 0,\ 0\le x\le\sqrt{\frac{y+1}{2}}\right\}\cup\left\{(x,y):0\le y\le 1,\ \sqrt{y}\le x\le\sqrt{\frac{y+1}{2}}\right\}$ は $D=\{(x,y):0\le x\le 1,\ 2x^2-1\le y\le x^2\}$ と表される事に注意すれば，

$$I=\int_0^1\left(\int_{2x^2-1}^{x^2}\sqrt{y-x^2+1}dy\right)dx=\frac{2}{3}\int_0^1\Big[(y-x^2+1)^{\frac{3}{2}}\Big]_{2x^2-1}^{x^2}dx=\frac{2}{3}\int_0^1(1-x^3)dx=\frac{1}{2}$$

□

**5.** 次の累次積分の積分順序を交換し，2 つの 2 重積分の和で表せ.

(1) $\displaystyle\int_0^1\left(\int_0^{x+1}f(x,y)dy\right)dx.$ (2) $\displaystyle\int_0^{\sqrt{2}}\left(\int_y^{\sqrt{4-y^2}}f(x,y)dx\right)dy$

【解答】 以下，求める積分を $I$，$I$ の積分領域を $D$ と記す．

(1) $D = \{(x,y) : 0 \le y \le 1,\ 0 \le x \le 1\} \cup \{(x,y) : 1 \le y \le 2,\ 1-y \le x \le 1$ より

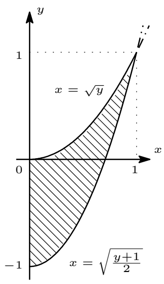
$$I = \int_0^1 \left( \int_0^1 f(x,y)dx \right) dy + \int_1^2 \left( \int_{1-y}^1 f(x,y)dx \right) dy.$$

(2) $D = \{(x,y) : 0 \le x \le \sqrt{2},\ 0 \le y \le x\} \cup \{(x,y) : \sqrt{2} \le x \le 2,\ 0 \le y \le \sqrt{4-x^2}\}$ より

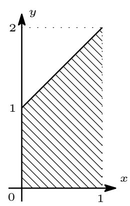
$$I = \int_0^{\sqrt{2}} \left( \int_0^x f(x,y)dy \right) dx + \int_{\sqrt{2}}^2 \left( \int_0^{\sqrt{4-x^2}} f(x,y)dy \right) dx.$$

□

4.(3)




5.(1)




5.(2)